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Minimarg and Maximarg Operators

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Abstract. Two operators on the set of n-person cooperative games are introduced, the minimarg operator and the maximarg operator. These operators can be seen as dual to each other. Some nice properties of these operators are given, and classes of games for which these operators yield convex (respectively, concave) games are considered. It is shown that, if these operators are applied iteratively on a game, in the limit one will yield a convex game and the other a concave game, and these two games will be dual to each other. Furthermore, it is proved that the convex games are precisely the fixed points of the minimarg operator and that the concave games are precisely the fixed points of the maximarg operator.

Key Words. Convex games, marginal contributions, duality, symmetric games.

1. Introduction

Shapley (Ref. 1) used the marginal contributions of a player in a cooperative game to define a map from the set of n-person cooperative games to R^n. By identifying in the obvious way the Shapley value of a game with an additive game, the Shapley value can be seen as an operator on the set of cooperative games. The fixed points of this operator are precisely the additive games. The underlying assumption in the definition of this operator is that all possible orders of formations of the grand coalition are equally likely. One of the operators to be defined in this paper will reflect a pessimistic view on part of each coalition with regard to the order of formation of the grand

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coalition. Each coalition believes that this will happen in a worst possible way for it. The other operator reflects an optimistic view on part of each coalition with regard to the order of formation of the grand coalition. Each coalition believes that this will happen in a best possible way for it. In Section 2, the operators will be defined and several properties of these operators will be investigated. It will be proved that, for some classes of games, one of these operators will yield convex games while the other will yield concave games. Furthermore, it will be shown that applying one of the operators iteratively on any game will yield a convex game in the limit, while doing this with the other operator will yield a concave game. In Section 3, it will be shown that a characterization of convex games as fixed points of one of the operators can be given while concave games can be characterized as the fixed points of the other.

2. Minimarg and Maximarg Operators

An n-person cooperative game is an ordered pair $\langle N, v\rangle$ where $N=\{1,2,\ldots,n\}$ is the finite set of players and v is the characteristic function which assigns to every subset of N a real number with the condition that $v(\emptyset)=0$. The set of subsets of N is denoted by 2^N. A subset of N is called a coalition. Whenever there can be no cause for confusion, the game $\langle N, v\rangle$ will be identified with the function v. The set of n-person games will be denoted by G^n.

A game $v\in G^N$ is said to be superadditive if

$$v(S\cup T)\geq v(S)+v(T), \qquad \text{for all } S, T\in 2^N \text{ with } S\cap T=\emptyset. \tag{1}$$

A game v is said to be subadditive if the reverse inequality holds in (1) and additive if equality holds.

A game v is said to be convex if

$$v(S\cup T)+v(S\cap T)\geq v(S)+v(T), \qquad \text{for all } S, T\in 2^N. \tag{2}$$

A game v is said to be concave if the reverse inequality holds in (2).

Let $v\in G^n$. The dual game v^* of v is defined by

$$v^*(S)=v(N)-v(N\backslash S), \qquad \text{for all } S\in 2^N. \tag{3}$$

Let $v\in G^n$. The core $C(v)$ of v is defined by

$$C(v):=\left\{x\in R^n \,\middle|\, \sum_{i\in N} x_i=v(N),\ \sum_{i\in S} x_i\geq v(S), \text{ for all } S\in 2^N\right\}.$$

A game $v \in G^n$ is said to be exact if

$$\text{for all } S \subset N, \text{ there exists an } x \in C(v) \text{ with } \sum_{j \in S} x_j = v(S).$$

Schmeidler (Ref. 2) noted that any convex game is exact.

Let $\sigma: N \to N$ be a permutation of N. We denote the set of permutations of N by Π. A permutation of N can be considered as defining an order in which members of N are entering a room. Every time a new member enters, a new coalition is formed until N itself is formed. If the entering order is according to σ, then $\sigma(i) = j$ means that $i \in N$ has the jth position in entering. For $S \in 2^N$, the set $\{\sigma(i) \mid i \in S\}$ is denoted by $\sigma(S)$. Let

$$P(\sigma, i) := \{j \in N \mid \sigma(j) < \sigma(i)\}.$$

Before i enters the coalition, $P(\sigma, i)$ has been formed. After is entrance, the coalition $P(\sigma, i) \cup \{i\}$ is formed. The marginal contribution of i, if the entering order is according to σ, is defined to be

$$m_v^\sigma(i) := v(P(\sigma, i) \cup \{i\}) - v(P(\sigma, i)).$$

For every $\sigma \in \Pi$, we can consider the marginal vector defined by σ with ith coordinate equal to $m_v^\sigma(i)$. The Weber set $W(v)$ of v is defined to be the convex hull of the $n!$ marginal vectors that we obtain in this way. Weber (Ref. 3) proved that, for any game $v \in G^n$, the core of v is a subset of the Weber set.

For every $\sigma \in \Pi$, an additive game m_v^σ is defined by

$$m_v^\sigma(S) = \sum_{i \in S} m_v^\sigma(i).$$

The Shapley value, when viewed as an operator on G^n, assigns to every game v the average of the additive marginal games m_v^σ. Let ϕ denote this operator. Then,

$$\phi(v) = (1/n!) \sum_{\sigma \in \Pi} m_v^\sigma.$$

So, the Shapley operator assigns to each game a linear combination of additive games. Here, two operators will be defined. One assigns to every game the minimum of the marginal games, while the other assigns to every game the maximum of the marginal games. Formally, the minimarg operator Mi is defined by

$$\mathrm{Mi}(v) := \min_{\sigma \in \Pi} m_v^\sigma, \qquad \text{for all } v \in G^n.$$

So, for any game $v \in G^n$, the game $\mathrm{Mi}(v)$ is defined by

$$\mathrm{Mi}(v)(S) := \min_{\sigma \in \Pi} m_v^\sigma(S), \qquad \text{for all } S \in 2^N.$$

The maximarg operator Ma is defined by

$$\mathrm{Ma}(v) := \max_{\sigma \in \Pi} m_v^\sigma, \qquad \text{for all } v \in G^n.$$

For any game $v \in G^n$, the game $\mathrm{Ma}(v)$ is defined by

$$\mathrm{Ma}(v)(S) = \max_{\sigma \in \Pi} m_v^\sigma(S), \qquad \text{for all } S \in 2^N.$$

The following intuitive explanation can be given for the games $\mathrm{Mi}(v)$ and $\mathrm{Ma}(v)$. Consider a game v where $v(S)$ reflects the profit that coalition S can achieve if it forms. The grand coalition is considered to be formed according to some order. Every member of the grand coalition receives his marginal contribution in this process. The worth of a coalition is the sum of what its members receive. A coalition S may consider a worst possible order, i.e., an order for which the sum of what its members receive is minimal. The worth of S in this case is $\mathrm{Mi}(v)(S)$. On the other hand, $\mathrm{Ma}(v)(S)$ is the worth of coalition S when the grand coalition is formed according to a best possible order for S. If v is a cost game, $\mathrm{Mi}(v)$ and $\mathrm{Ma}(v)$ reverse roles.

In the following theorem, some properties of these operators are given.

Theorem 2.1. Let $v \in G^n$. Then:

(i) $\mathrm{Mi}(v)(S) \leq v(S)$ and $\mathrm{Ma}(v)(S) \geq v(S)$, for all $S \subset N$;

$\mathrm{Mi}(v)(N) = \mathrm{Ma}(v)(N) = v(N)$;

(ii) $\mathrm{Mi}(v)$ is superadditive and $\mathrm{Ma}(v)$ is subadditive;
(iii) $W(v) \subset C(\mathrm{Mi}(v))$;
(iv) $\mathrm{Mi}(v)$ is exact;
(v) $\mathrm{Mi}(\mathrm{Ma})$ is a continuous operator on G^n.

Proof. (i) and (ii) are evident from the definition of Mi and Ma.

For (iii), note that, for each $\sigma \in \Pi$, we have $m_v^\sigma(S) \geq \mathrm{Mi}(v)(S)$, for each $S \subset N$, and $m_v^\sigma(N) = v(N)$. Thus, it follows that $m_v^\sigma \in C(\mathrm{Mi}(v))$, for all $\sigma \in \Pi$, and hence $W(v) \subset C(\mathrm{Mi}(v))$.

(iv) is proved by taking $\sigma \in \Pi$ such that $\mathrm{Mi}(v)(S) = m_v^\sigma(S)$ and by noting that, from (iii), it follows that $m_v^\sigma \in C(\mathrm{Mi}(v))$.

Finally, (v) follows from the fact that $\mathrm{Mi}(\mathrm{Ma})$ is the minimum (maximum) of a finite number of linear operators on G^n. □

The following theorem states that the minimarg and the maximarg operator can be seen as dual to each other.

Theorem 2.2. Let $v \in G^n$. Then, $\mathrm{Mi}(v) = (\mathrm{Ma}(v))^*$.

Proof. For $S \in 2^N$, let $\sigma(S) \in \Pi$ be such that $\mathrm{Mi}(v)(S) = m_v^{\sigma(S)}(S)$. Then, $\mathrm{Ma}(v)(N \backslash S) = m_v^{\sigma(S)}(N \backslash S)$ by the definition of $\mathrm{Ma}(v)$ and the fact that $m_v^\sigma(S) + m_v^\sigma(N \backslash S) = v(N)$, for all $\sigma \in \Pi$. It follows that

$$\begin{aligned}(\mathrm{Ma}(v))^*(S) &= \mathrm{Ma}(v)(N) - \mathrm{Ma}(v)(N \backslash S)\\ &= v(N) - m_v^{\sigma(S)}(N \backslash S) = m_v^{\sigma(S)}(S) = \mathrm{Mi}(v)(S).\end{aligned}$$

Hence, $\mathrm{Mi}(v) = (\mathrm{Ma}(v))^*$. □

In the following theorem, it is shown that two games that are dual to each other have the same image under the maximarg (minimarg) operator.

Theorem 2.3. Let $v, w \in G^n$ with $w = v^*$. Then,

(i) $\mathrm{Ma}(w) = (\mathrm{Mi}(v))^* = \mathrm{Ma}(v)$,
(ii) $\mathrm{Mi}(w) = (\mathrm{Ma}(v))^* = \mathrm{Mi}(v)$.

Proof. (i) For a permutation σ of N, let $F(\sigma, i)$ denote the set of followers of i with respect to σ, that is,

$$F(\sigma, i) := \{j \in N \mid \sigma(j) > \sigma(i)\}.$$

Then, for all $S \subset N$, we have

$$m_w^\sigma(S) = \sum_{i \in S} (w(P_i^\sigma \cup i) - w(P_i^\sigma)) = \sum_{i \in S} (v(F_i^\sigma \cup i) - v(F_i^\sigma)) = m_v^{\bar\sigma}(S).$$

Here, the second equality follows from the fact that $w = v^*$. The permutation $\bar\sigma$ is obtained by reversing the order defined by σ. So, formally we have

$$\bar\sigma(i) = n + 1 - \sigma(i).$$

The proof is completed by observing that

$$\mathrm{Ma}(w)(S) = \max_{\sigma \in \Pi} m_w^\sigma(S) = \max_{\bar\sigma \in \Pi} m_v^{\bar\sigma}(S) = \mathrm{Ma}(v)(S) = (\mathrm{Mi}(v))^*(S),$$

where the last equality follows from Theorem 2.2.

(ii) The proof is similar to that of (i). □

Using Theorem 2.3, it can be proved that if the minimarg and the maximarg operators are used iteratively on a game the result depends on which one has been applied last.

Theorem 2.4. Let $v \in G^n$. Then, $\mathrm{Mi}(\mathrm{Ma}(v)) = \mathrm{Mi}(\mathrm{Mi}(v))$ and $\mathrm{Ma}(\mathrm{Mi}(v)) = \mathrm{Ma}(\mathrm{Ma}(v))$.

Proof. Using Theorem 2.2 and Theorem 2.3, we obtain

$$\mathrm{Mi}(\mathrm{Ma}(v)) = \mathrm{Mi}((\mathrm{Mi}(v))^*) = \mathrm{Mi}(\mathrm{Mi}(v)).$$

The second part of the theorem can be proved similarly. □

In a straightforward way, the proof of Theorem 2.4 can be generalized to more than two iterations by subsequently using Theorems 2.2, 2.3, and 2.4. For example, the proof that

$$\mathrm{Ma}(\mathrm{Mi}(\mathrm{Ma}(v))) = \mathrm{Ma}(\mathrm{Ma}(\mathrm{Ma}(v)))$$

runs as follows:

$$\begin{aligned}\mathrm{Ma}(\mathrm{Mi}(\mathrm{Ma}(v))) &= \mathrm{Ma}(\mathrm{Mi}(\mathrm{Mi}(v))) = \mathrm{Ma}(\mathrm{Mi}((\mathrm{Ma}(v))^*)) \\ &= \mathrm{Ma}((\mathrm{Ma}((\mathrm{Ma}(v))^*))_*) = \mathrm{Ma}((\mathrm{Ma}(\mathrm{Ma}(v)))^*) \\ &= \mathrm{Ma}(\mathrm{Ma}(\mathrm{Ma}(v))).\end{aligned}$$

Here, the first equality follows from Theorem 2.4, the second and third equalities follow from Theorem 2.2, and the fourth and fifth equalities follow from Theorem 2.3.

For a game v with three or less players, $\mathrm{Mi}(v)$ is convex and $\mathrm{Ma}(v)$ is concave as the following proposition states.

Proposition 2.1. Let $n \leq 3$ and $v \in G^n$. Then, $\mathrm{Mi}(v)$ is convex and $\mathrm{Ma}(v)$ is concave.

Proof. For $n=1$, it is evident that the statement of the proposition is true. For $n=2$, the result follows from the fact that $\mathrm{Mi}(v)$ is superadditive and $\mathrm{Ma}(v)$ is subadditive.

Let $n=3$, say $N=\{i,j,k\}$. To prove that $\mathrm{Mi}(v)$ is convex, the only kind of inequality that has to be proved is

$$\mathrm{Mi}(v)(\{i,j\}) + \mathrm{Mi}(v)(\{i,k\}) \leq \mathrm{Mi}(v)(\{i,j,k\}) + \mathrm{Mi}(v)(\{i\}),$$

because the other inequalities are either obvious or follow from the superadditivity of $\mathrm{Mi}(v)$. Assume that

$$\mathrm{Mi}(v)(\{i\}) = m_v^\sigma(i).$$

Then,

$$\begin{aligned}\mathrm{Mi}(v)(\{i\}) + \mathrm{Mi}(v)(\{i,j,k\}) &= m_v^\sigma(i) + m_v^\sigma(\{i,j,k\}) \\ &= m_v^\sigma(\{i,j\}) + m_v^\sigma(\{i,k\}) \\ &\geq \mathrm{Mi}(v)(\{i,j\}) + \mathrm{Mi}(v)(\{i,k\}).\end{aligned}$$

It follows that $\mathrm{Mi}(v)$ is convex. The concavity of $\mathrm{Ma}(v)$ follows from Theorem 2.2 and the well-known fact that the dual game of a convex game is concave. □

The following example shows that, in general, $\mathrm{Mi}(v)$ need not be convex and $\mathrm{Ma}(v)$ need not be concave.

Example 2.1. Let $N=\{1, 2, 3, 4\}$. Let $v(\{i\})=0$, for all $i\in N$. Let

$$v(\{1, 2\})=v(\{1, 3\})=v(\{1, 4\})=v(\{2, 4\})=v(\{3, 4\})=1,$$

$$v(\{2, 3\})=1/2, \qquad v(\{1, 2, 3\})=v(\{1, 3, 4\})=v(\{1, 2, 4\})=2,$$

$$v(\{2, 3, 4\})=7/2, \qquad v(N)=9/2.$$

Straightforward calculation shows that $\mathrm{Mi}(v)(\{i\})=0$, for all $i\in N$,

$$\mathrm{Mi}(v)(\{1, 2\})=\mathrm{Mi}(v)(\{1, 3\})=\mathrm{Mi}(v)(\{1, 4\})$$
$$=\mathrm{Mi}(v)(\{2, 4\})=\mathrm{Mi}(v)(\{3, 4\})=1,$$

$$\mathrm{Mi}(v)(\{2, 3\})=1/2, \qquad \mathrm{Mi}(v)(\{1, 2, 3\})=3/2,$$

$$\mathrm{Mi}(v)(\{1, 2, 4\})=\mathrm{Mi}(v)(\{1, 3, 4\})=2,$$

$$\mathrm{Mi}(v)(\{2, 3, 4\})=3, \qquad \mathrm{Mi}(v)(N)=9/2.$$

It follows that $\mathrm{Mi}(v)$ is not convex, because

$$\mathrm{Mi}(v)(\{1, 2\})+\mathrm{Mi}(v)(\{1, 3\})$$
$$=1+1=2>3/2+0=\mathrm{Mi}(v)(\{1, 2, 3\})+\mathrm{Mi}(v)(\{1\}).$$

Since $\mathrm{Mi}(v)$ is the dual game of $\mathrm{Ma}(v)$, and the dual game of a concave game is convex, it also follows that $\mathrm{Ma}(v)$ is not concave.

It is possible to apply the minimarg operator iteratively to a game $v\in G''$. As we show in the following theorem, the sequence of games generated in this way converges to a convex game.

Theorem 2.5. Let $v\in G''$. For $k=1, 2, \ldots$, define the games v^k by $v^1=v$ and $v^k=\mathrm{Mi}(v^{k-1})$. Then, $\lim_{k\to\infty} v^k$ exists and is convex.

Proof. Let $S\subset N$. From property (1) in Theorem 2.1, it follows that $v^1(S)\geq v^2(S)\geq \ldots$. Furthermore, for any superadditive game w, $m_w^\sigma(i)\geq w(\{i\})$, for all $\sigma\in\Pi$, and therefore

$$\mathrm{Mi}(w)(S)\geq \sum_{i\in S} w(\{i\}).$$

Property (ii) of Theorem 2.1 implies that v^k is superadditive for $k=2, 3, \ldots$. Combining this with the previous statement yields

$$v^k(S) \geq \sum_{i \in S} \mathrm{Mi}(v)(\{i\}), \qquad \text{for } k=2, 3, \ldots.$$

Hence, $\lim_{k\to\infty} v^k(S)$ exists. Denote the limit by $z(S)$. Then,

$$z = \lim_{k\to\infty} v^k.$$

Now,

$$z = \lim_{k\to\infty} v^k = \lim_{k\to\infty} \mathrm{Mi}(v^k) = \mathrm{Mi}(\lim_{k\to\infty} v^k) = \mathrm{Mi}(z).$$

So, z is a fixed point of Mi, and by the result of Theorem 3.1 in the next section, it follows that z is a convex game. □

Note that, in a similar way, it can be proved that applying Ma iteratively to a game v yields a sequence of games that converges to a concave game. Using Theorems 2.2 and 2.3, it can be shown that the games obtained in each iteration by applying Mi and Ma will be each other's dual. Since taking the dual game of a game is a continuous operation on G^n, it follows that the games that are obtained in the limit will also be each other's dual.

Theorem 2.5 raises the following interesting question. For which games is it sufficient to perform only a finite number of iterations in order to obtain a convex game? From Proposition 2.1, it follows that, for $n \leq 3$, at most one iteration is needed. The following theorem gives another class of games for which this is the case. A game $v \in G^n$ is said to be symmetric if

$$v(S) = v(T), \qquad \text{for every } S, T \in 2^N, \text{ with } |S| = |T|.$$

Here, $|S|$ is the number of members of S. The following theorem states that the image of a symmetric game under the minimarg operator is convex and the image of a symmetric game under the maximarg operator is concave.

Theorem 2.6. Let $v \in G^n$ be symmetric. Then, $\mathrm{Mi}(v)$ is convex and $\mathrm{Ma}(v)$ is concave.

Proof. Let $v \in G^n$ be symmetric. Then, there exists an $f: \{0, 1, 2, \ldots, n\} \to R$, with

$$v(S) = f(|S|), \qquad \text{for each } S \in 2^N.$$

Let $d_1, d_2, \ldots, d_n$ be the sequence of differences

$$\{f(k)-f(k-1) \mid k=1, 2, \ldots, n\},$$

placed in increasing order, i.e.,

$$d_1 \le d_2 \le \cdots \le d_n.$$

From the definition of $\mathrm{Mi}(v)$ it follows that, for each $S \in 2^N$,

$$\mathrm{Mi}(v)(S) = \sum_{i=1}^{s} d_i, \qquad \text{if } |S| = s.$$

Then, for $S \subset T \subset N \setminus \{i\}$, we have

$$\begin{aligned}&\mathrm{Mi}(v)(S \cup \{i\}) - \mathrm{Mi}(v)(S)\\ &= d_{|S|+1} \le d_{|T|+1} = \mathrm{Mi}(v)(T \cup \{i\}) - \mathrm{Mi}(v)(T).\end{aligned}$$

Thus, $\mathrm{Mi}(v)$ is a convex game, and by Theorem 2.2 it follows that $\mathrm{Ma}(v)$ is a concave game. □

An example of a game that is not symmetric, but for which it is also true that a finite number of iterations will yield a convex game, is given by Example 2.1.

Example 2.2. After one iteration, we obtained the game $\mathrm{Mi}(v) = v^2$, with $\mathrm{Mi}(v)(\{i\}) = 0$, for all $i \in N$,

$$\begin{aligned}\mathrm{Mi}(v)(\{1, 2\}) &= \mathrm{Mi}(v)(\{1, 3\}) = \mathrm{Mi}(v)(\{1, 4\})\\ &= \mathrm{Mi}(v)(\{2, 4\}) = \mathrm{Mi}(v)(\{3, 4\}) = 1,\end{aligned}$$

$$\mathrm{Mi}(v)(\{2, 3\}) = 1/2, \qquad \mathrm{Mi}(v)(\{1, 2, 3\}) = 3/2,$$

$$\mathrm{Mi}(v)(\{1, 2, 4\}) = \mathrm{Mi}(v)(\{1, 3, 4\}) = 2,$$

$$\mathrm{Mi}(v)(\{2, 3, 4\}) = 3, \qquad \mathrm{Mi}(v)(N) = 9/2.$$

After the second iteration, we obtain the game $v^3 = \mathrm{Mi}(v^2)$ given by $v^3(i) = 0$, for all $i \in N$,

$$v^3(\{1, 2\}) = v^3(\{1, 3\}) = v^3(\{2, 3\}) = 1/2,$$

$$v^3(\{1, 4\}) = v^3(\{2, 4\}) = v^3(\{3, 4\}) = 1, \qquad v^3(\{1, 2, 3\}) = 3/2,$$

$$v^3(\{1, 2, 4\}) = v^3(\{1, 3, 4\}) = v^3(\{2, 3, 4\}) = 2, \qquad v^3(N) = 9/2.$$

Straightforward verification shows that v^3 is a convex game.

3. Convex Games

In this section, a characterization of convex games and concave games will be given using the minimarg and the maximarg operator. Several characterizations of convex games have been given in the literature; see Shapley (Ref. 4), Rosenmüller (Ref. 5), Ichiishi (Ref. 6), Kikuta (Ref. 7), and Monderer, Samet, and Shapley (Ref. 8). Two of these characterizations will be mentioned here. The first one is due to Shapley (Ref. 4) and states that a game is convex if and only if the marginal contribution of a coalition does not decrease when the coalition that it joins becomes larger. Formally, a game $v \in G^n$ is convex if and only if

$$v(S \cup R) - v(S) \leq v(T \cup R) - v(T),$$

$$\text{for all } S, T, R \in 2^N \text{ with } S \subset T \subset N \backslash R.$$

The second characterization states that a game is convex if and only if the Weber set of the game is equal to the core of the game. The "if" part is due to Ichiishi (Ref. 6) and the "only if" part is due to Shapley (Ref. 4). Let m_v^σ denote the vector in R^n with ith coordinate equal to $m_v^\sigma(i)$. Then,

$$\text{a game } v \in G^n \text{ is convex if and only if } m_v^\sigma \in C(v) \text{ for all } \sigma \in \Pi.$$

A game $v \in G^n$ is called a fixed point of the operator Mi if $\mathrm{Mi}(v) = v$. The following theorem states that the convex games are precisely the fixed points of Mi. This result follows easily from the characterization of convex games of Ichiishi and Shapley.

Theorem 3.1. Let $v \in G^n$. Then, $\mathrm{Mi}(v) = v$ if and only if v is convex.

Proof. Since $\mathrm{Mi}(v)(S) \leq v(S)$, for all $S \in 2^N$, and $\mathrm{Mi}(v)(N) = v(N)$, the statement $\mathrm{Mi}(v) = v$ is equivalent to $m_v^\sigma \in C(v)$, for all $\sigma \in \Pi$. From the second characterization given above, it follows that this is equivalent to v being convex. □

A similar proof can be given for the fact that a game v is a fixed point of the operator Ma if and only if v is concave.

Theorem 2.3 and Theorem 3.1 lead to the following corollaries.

Corollary 3.1. Let v be a concave (convex) game. Then, $\mathrm{Mi}(v)$ ($\mathrm{Ma}(v)$) is convex (concave).

Corollary 3.2. Let $v \in G^n$. Then $\mathrm{Mi}(v) = v^*$ ($\mathrm{Ma}(v) = v^*$) if and only if v is concave (convex).

Remark 3
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Remark 3.1. Rosenmüller (Ref. 5) also uses marginal contributions to give a representation of convex games. However, he looks at the marginal contributions minus a constant and takes the maximum instead of the minimum. A result paralleling the one of Theorem 3.1 in this paper is obtained by Schmeidler (Ref. 2), who defines the exact envelope of a game and proves that a game is exact iff it is equal to its exact envelope.

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